Method of characteristics

<u>Main idea</u>: switch coordinates from (x,y) to new coordinates (s,t) so that along a curve of constant t, the solution is governed by an ODE with only derivatives in s. Such a curve is called a <u>characteristic</u>.

<u>Ex 1</u> $\quad u_x + y u_y = 0, \qquad u(0,y) = y^3$

$$\Rightarrow \begin{pmatrix} 1 \\ y \end{pmatrix} \cdot \nabla u = 0$$

directional derivative.

We need to find a curve C (actually a family of curves, one for each t) such that the tangent to C is parallel to $\begin{pmatrix} 1 \\ y \end{pmatrix}$ always. Then u is constant along this curve:

$$\left.\frac{du}{ds}\right|_C = 0$$

s: how far along the curve
t: which curve

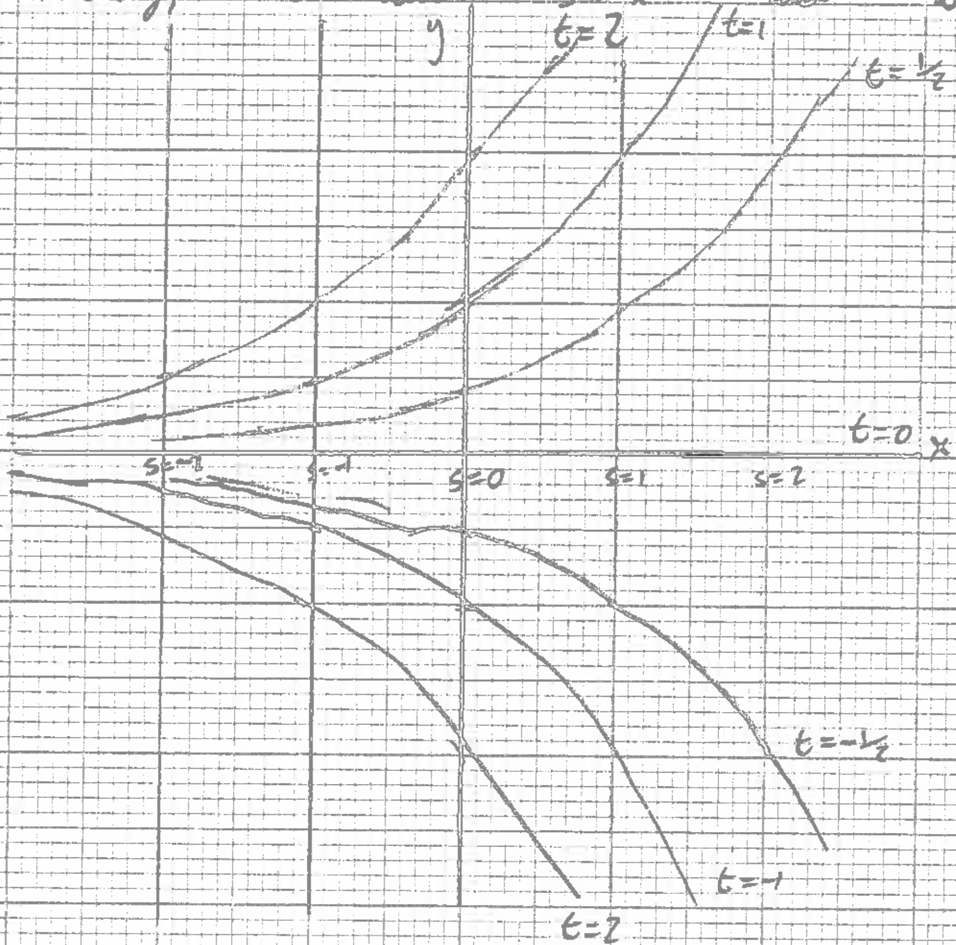
Say C is parameterized by s: then

$$\frac{dx}{ds} = 1 \quad \text{and} \quad \frac{dy}{ds} = y$$

$$\Rightarrow \quad x = s, \qquad y = t e^s$$

arbitrarily choosing $s = 0$ to correspond to $x = 0$.
This is useful because our ICs are on $x = 0$.

Inverting, we have $s = x$ and $t = ye^{-x}$



And here the characteristics fill the plane. (They don't always, though — in which case the solution would only apply to part of the plane)

Along each characteristic, $\frac{du}{ds} = 0$ so ~~$u(s,t) = u(0,t)$~~

not different function

Say $u(x,y) = V(s,t)$ (same quantity, but expressed as a function of s,t)

$= v(0,t)$

~~$u(0,y)$~~

$= t^3 = y^3 e^{-3x}$.

Ex 2 $\qquad u_x + y u_y = -\lambda u \qquad u(0,y) = y^3$

Same LHS so the characteristics are the same (they depend on the operator).

But this time $\left.\frac{du}{ds}\right|_c = \left.\frac{\partial v}{\partial s}\right|_t = -\lambda u$

$$\text{so} \quad v(s,t) = v(0,t)\, e^{-\lambda s}$$

$$\Rightarrow u(x,y) = t^3 e^{-\lambda s}$$

$$= y^3 e^{-3x - \lambda x}$$

$$= y^3 e^{-(3+\lambda)x}.$$

Ex 3 $\qquad y u_x + x u_y = 0 \qquad u(0,y) = e^{-y^2}$

$$\begin{pmatrix} y \\ x \end{pmatrix} \cdot \nabla u = 0$$

so along curves with tangent $\begin{pmatrix} y \\ x \end{pmatrix}$, u is constant.

Along such a curve,

$$\frac{dx}{ds} = y \quad , \quad \frac{dy}{ds} = x \qquad t^2 = y^2 - x^2$$

so $\quad x = t \sinh s \quad , \quad y = t \cosh s$ — and $t = y$ on $s = 0$

taking these solutions so that $s=0$ on the initial data, $x=0$.

Since u doesn't depend on s,

$$u(x,y) = v(s,t) = v(0,t)$$

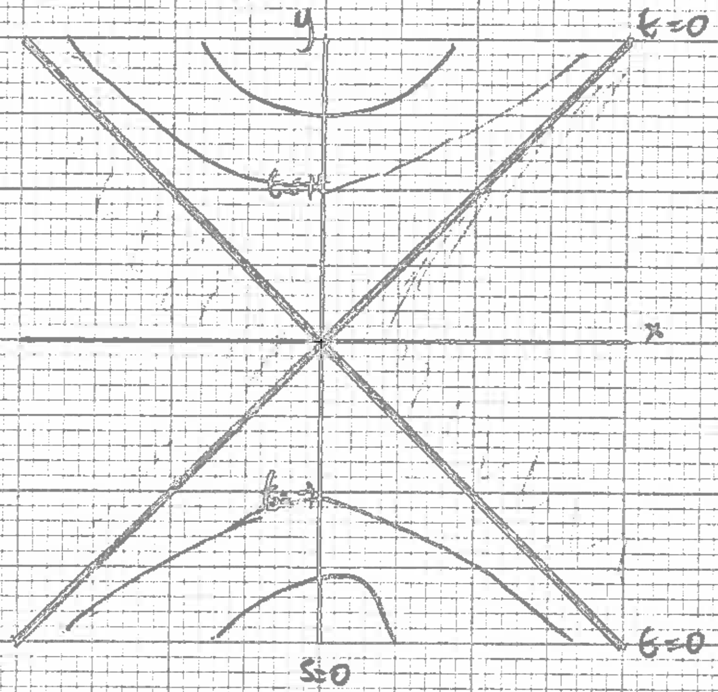
$$= u(0,y)$$

$$= e^{-y^2} = e^{-t^2}$$

$$\text{so} \quad v(s,t) = e^{-t^2}$$

$$u(x,y) = e^{-(y^2 - x^2)}.$$

Note that these characteristics emanating from $x=0$, $s=0$ do not fill the plane.



MoC for 2^{nd} order hyperbolic equations

Again the idea is to reduce the problem to ode, as much as possible, or to a first order eqn. Again we have to switch coordinates.

Basic example

$$\cancel{u_{tt} - c^2 u_{xx} = 0}$$

$$u_{tt} - c^2 u_{xx} = 0$$

Take $\quad \xi = x - ct, \qquad \eta = x + ct$

$$x = \frac{1}{2}(\xi + \eta)$$

$$t = \frac{1}{2c}(-\xi + \eta).$$

Then $$\cancel{\frac{\partial}{\partial x} = \frac{1}{2}\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)}$$

$$\cancel{\frac{\partial^2}{\partial x^2} = \frac{1}{4}\left(\frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right)}$$

$$\cancel{\frac{\partial^2}{\partial t^2} = \frac{1}{4c^2}\left(\frac{\partial^2}{\partial \xi^2} - 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right)}$$

$$\left.\frac{\partial}{\partial x}\right|_t = \left.\frac{\partial}{\partial \xi}\right|_\eta + \left.\frac{\partial}{\partial \eta}\right|_\xi$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}$$

$$\frac{1}{c^2}\frac{\partial}{\partial t^2} = \;'' \;-\; '' \;+\; ''$$

and so $$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

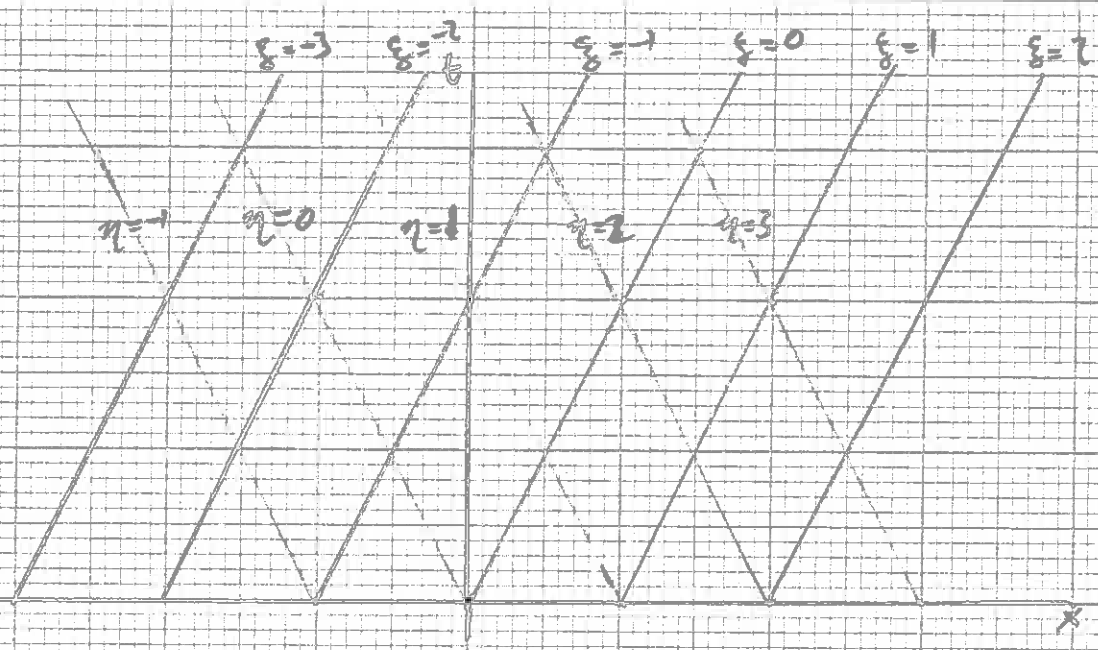
Reduced to having only 1^{st} derivatives: this is the <u>canonical</u> form.

Integrating w.r.t. η:
$$\frac{\partial u}{\partial \xi} = f'(\xi)$$

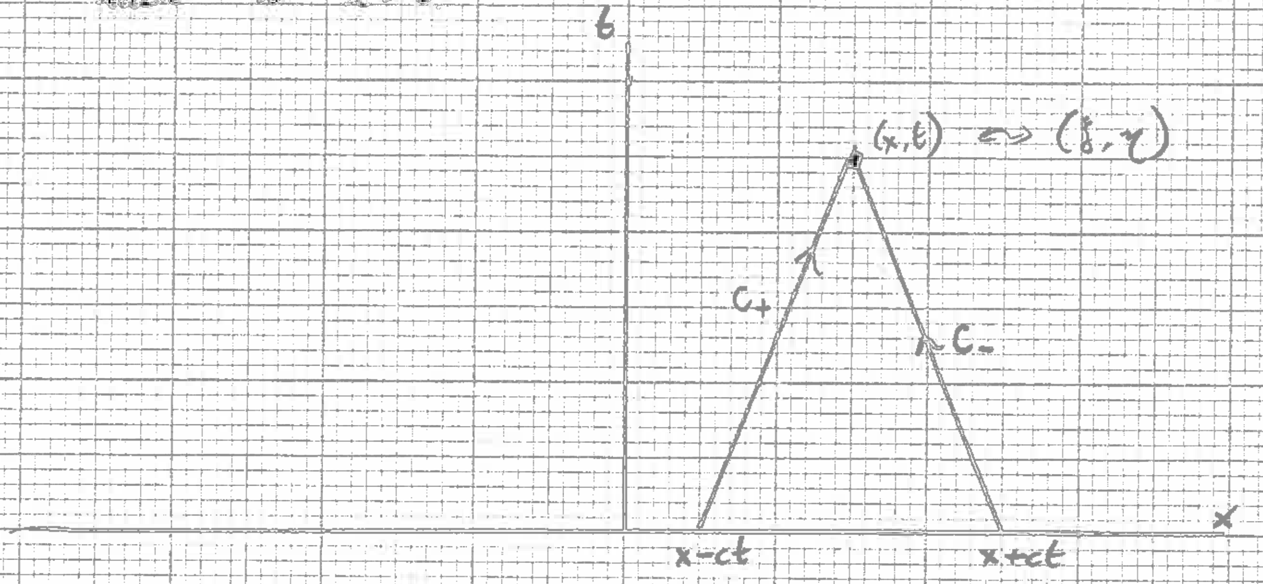
depends only on ξ

Take f' and set f for notational convenience

Integrating again: w.r.t. ξ:
$$u = f(\xi) + g(\eta)$$
$$= f(x-ct) + g(x+ct)$$



To get $u(x,t)$ for initial conditions ~~$u(x,t)$~~ $u(x,0)$,
follow both the C_+ characteristic $\xi = \text{const}$.
and the C_- characteristic $\eta = \text{const}$
back to $t=0$



$$u(x,t) = f(\xi) + g(\eta)$$
$$= f(x-ct) + g(x+ct)$$

where f, g depend on the ICs for u and u_t.

In this basic example, all the characteristics are straight lines. ~~But~~ It can be shown that for hyperbolic equations we can find pairs of families of characteristics,⊛ but in general they will not be straight.

⊛ This is the definition of hyperbolicity.

<u>Example:</u> ~~Chaplygin's equation~~ <u>Tricomi's equation</u> (slightly different notation from ex. sheet)

$$u_{tt} - \frac{1}{x} u_{xx} = 0$$

For $x>0$:

"wave equation but with variable wave speed"

(divergent wave speed at $x=0$, where the eqn becomes parabolic: information can travel infinitely fast in $x=0$)

Quantity: $\frac{\partial^2 u}{\partial \xi \partial \eta} = \ldots$ (hopefully 0)

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}$$

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial \xi}{\partial x}\right)^2 \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}\frac{\partial^2}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial x}\right)^2 \frac{\partial^2}{\partial \eta^2}$$

$$+ \left(1^{st} \text{ order derivatives}\right)$$

$$\frac{\partial^2}{\partial t^2} = \left(\frac{\partial \xi}{\partial t}\right)^2 \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial \xi}{\partial t}\frac{\partial \eta}{\partial t}\frac{\partial^2}{\partial \xi \partial \eta} +$$

$$+ \left(\frac{\partial \eta}{\partial t}\right)^2 \frac{\partial^2}{\partial \eta^2} + \left(1^{st} \text{ order derivatives}\right)$$

So $$\frac{\partial^2}{\partial t^2} - \frac{1}{x}\frac{\partial^2}{\partial x^2} = \left[\left(\frac{\partial \xi}{\partial t}\right)^2 - \frac{1}{x}\left(\frac{\partial \xi}{\partial x}\right)^2\right]\frac{\partial^2}{\partial \xi^2} + \ldots$$

To make $\partial^2/\partial\xi^2$ and $\partial^2/\partial\eta^2$ terms vanish

we want $\left(\frac{\partial\xi}{\partial t}\right)^2 = \frac{1}{x}\left(\frac{\partial\xi}{\partial x}\right)^2$, ditto with η.

$$\frac{\partial\xi}{\partial t} = -\frac{1}{x^{1/2}}\frac{\partial\xi}{\partial x}, \quad \text{or } + \text{ for } \eta$$

$$\Rightarrow \begin{pmatrix} 1 \\ +x^{1/2} \end{pmatrix} \cdot \begin{pmatrix} \partial\xi/\partial x \\ \partial\xi/\partial t \end{pmatrix} = \begin{pmatrix} 1 \\ +x^{1/2} \end{pmatrix} \cdot \nabla\xi = 0$$

This is just the 1st order problem!

So ξ is constant along curves with tangent $\begin{pmatrix} 1 \\ +x^{1/2} \end{pmatrix}$

$$\frac{dx}{ds} = 1, \qquad \frac{dt}{ds} = +x^{+1/2}$$

$$\Rightarrow \quad x = x_0 + s, \qquad \frac{dt}{ds} = +(x_0+s)^{1/2}$$

$$t = +\frac{2}{3}(x_0+s)^{3/2} = +\frac{2}{3}x^{3/2}$$

Or:

$$\frac{dx}{dt} = +x^{-1/2} \qquad \text{or } - \text{ for } \eta$$

$$\Rightarrow \quad x^{1/2}\,dx = dt$$

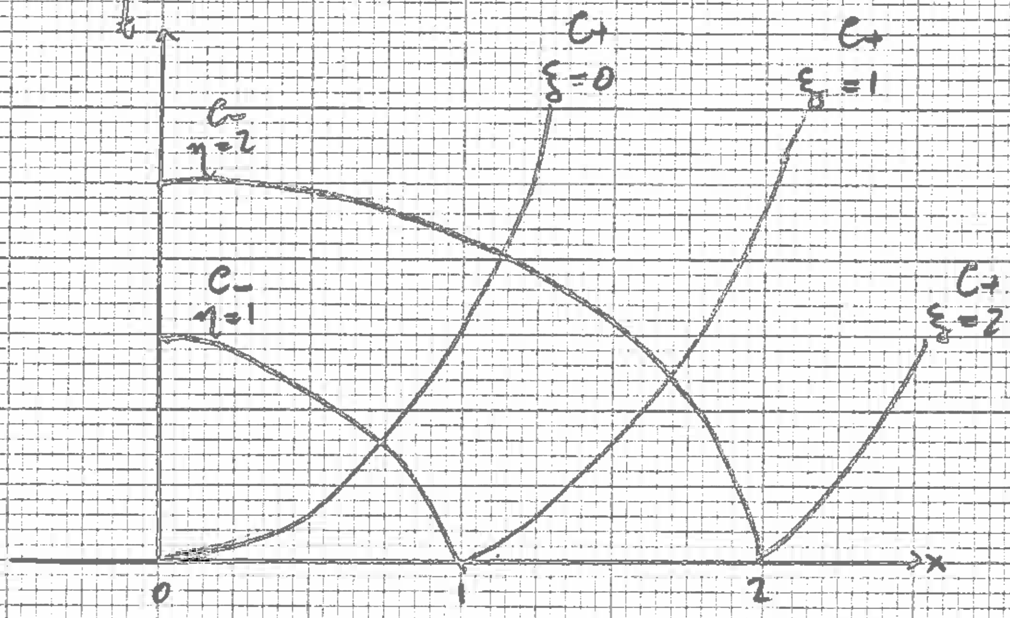
$$\Rightarrow \quad x^{3/2} - x_0^{3/2} = \frac{3}{2}t$$

$$x = \left(x_0^{3/2} + \frac{3}{2}t\right)^{2/3}$$

These are the C_+ characteristics, along which ξ is constant. We identify $\xi = x_0$

so that $\xi = \left(x^{3/2} - \frac{3}{2}t\right)^{2/3}$ right-travelling since $dx/dt > 0$

likewise $\eta = \left(x^{3/2} + \frac{3}{2}t\right)^{2/3}$ $\frac{dx}{dt} < 0$



The characteristics are curved, but again we follow the $C_\pm$ characteristics (on which ξ or η resp. are constant) to study the behaviour of u.
Note that u might not be constant along those characteristics — it depends on the 1st order terms. (†)
So the solution is NOT $u = f(\xi) + g(\eta)$.

(*) Example: Shallow water (Saint Venant) equations
used for modelling channel flow (eg. rivers) as well as weather systems (depth of atmosphere << horizontal lengthscales, so "shallow")

$$h_t + u h_x + h u_x = 0$$
$$u_t + u u_x + g h_x = 0.$$

(†) As in the 1st order case, the shapes of the characteristics are determined by the (highest-order) differential operators. The behaviour of u along those curves depends on the RHS — cf. ex. 1 vs. ex. 2.

What are the characteristics $C_\pm$ (curves, not coordinates!) for the shallow water system?

Idea: look for <u>Riemann invariants</u> $R_\pm$ that are constant, or at worst governed by 1st order o.d.es, along these curves.

$$\left.\frac{dR_+}{ds}\right|_{C_+} = \underline{n}\cdot\underline{\nabla}R_+ = 0$$

or maybe some fn of R_+

$$\underline{n} = \begin{pmatrix} dt/ds|_{C_+} \\ dx/ds|_{C_+} \end{pmatrix} \qquad \underline{\nabla} = \begin{pmatrix} \partial/\partial t \\ \partial/\partial x \end{pmatrix}$$

Ansatz: think of $V = \left.\frac{dx}{dt}\right|_{C_+}$ as some sort of wave speed. (Not necessarily constant!)

SWE: $$\frac{\partial}{\partial t}\begin{pmatrix} h \\ u \end{pmatrix} + \begin{pmatrix} u & h \\ g & u \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} h \\ u \end{pmatrix} = 0$$

So along a characteristic with $\frac{dx}{dt} = V$,

we have $$\left.\frac{d}{ds}\right|_{C_+} = \frac{dt}{ds}\frac{\partial}{\partial t} + \frac{dx}{ds}\frac{\partial}{\partial x} = 0$$

assuming conservation

$$\Rightarrow \frac{\partial}{\partial t} = -\frac{dx}{dt}\frac{\partial}{\partial x} = -V\frac{\partial}{\partial x}$$

So: $$\begin{bmatrix} u-V & h \\ g & u-V \end{bmatrix}\frac{d}{ds}\begin{pmatrix} h \\ u \end{pmatrix} = 0$$

Need to choose c s.t. $\begin{bmatrix} M \end{bmatrix}$ has a kernel

— this is an eigenvalue problem.

$$\det M = (u - V)^2 - gh = 0$$

$$\Rightarrow \quad V = u \pm (gh)^{1/2}$$

$$= \left.\frac{dx}{dt}\right|_{C_\pm} \quad \text{with } + \text{ for } C_+ \text{ characteristics,}$$
$$\text{and } - \text{ " } C_- \text{ " .}$$

N.B. $c = (gh)^{1/2}$ is the wave speed if $u = 0$;
For $u \neq 0$, $V = u \pm c$ is boosted.

So, $$M = \begin{bmatrix} \mp c & h \\ g & \mp c \end{bmatrix} = \begin{bmatrix} -(gh)^{1/2} & h \\ g & -(gh)^{1/2} \end{bmatrix} \quad \text{for } C_+$$

and its corresponding left eigenvector is $\cancel{\begin{pmatrix} h^{1/2} \\ g^{1/2} \end{pmatrix}}$

$$\begin{pmatrix} g^{1/2} & h^{1/2} \end{pmatrix}$$

so
$$g^{1/2}\frac{dh}{ds} + h^{1/2}\frac{du}{ds} = 0$$

$$= 2 g^{1/2} h^{1/2} \frac{d(h^{1/2})}{ds} + h^{1/2}\frac{du}{ds}$$

$$= h^{1/2}\frac{d}{ds}\left(u + 2(gh)^{1/2}\right) = 0$$

so $R_+ = u + 2(gh)^{1/2} = u + 2c$ is conserved along C_+ characteristics.

likewise $R_- = u - 2c$ is conserved along C_- characteristics.

⊛ Burger's equation, shocks and rarefactions (ill-posed nonlinear eq.)

$$u_t + u u_x = 0$$

$$\Rightarrow \begin{pmatrix} 1 \\ u \end{pmatrix} \cdot \begin{pmatrix} \partial u/\partial t \\ \partial u/\partial x \end{pmatrix} = 0$$

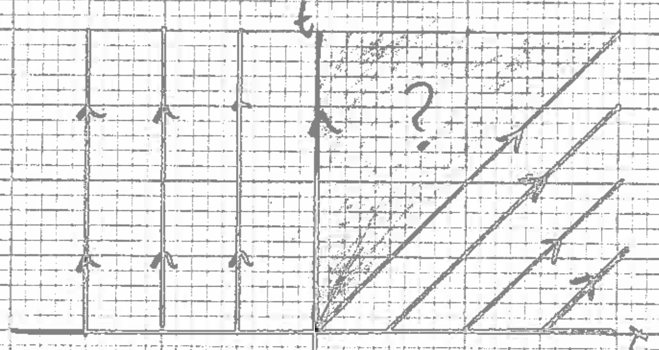
$\Rightarrow$ $u = \text{const}$ along characteristics $\dfrac{dx}{dt} = u$

$\therefore$ characteristics are straight lines. Higher $u \Rightarrow$ faster waves

Behaviour depends on ICs.

~~Rarefaction waves~~

Rarefaction waves: Consider $u(x,0) = \begin{cases} 0 & x<0 \\ 1/2 & x=0 \\ 1 & x>0 \end{cases}$

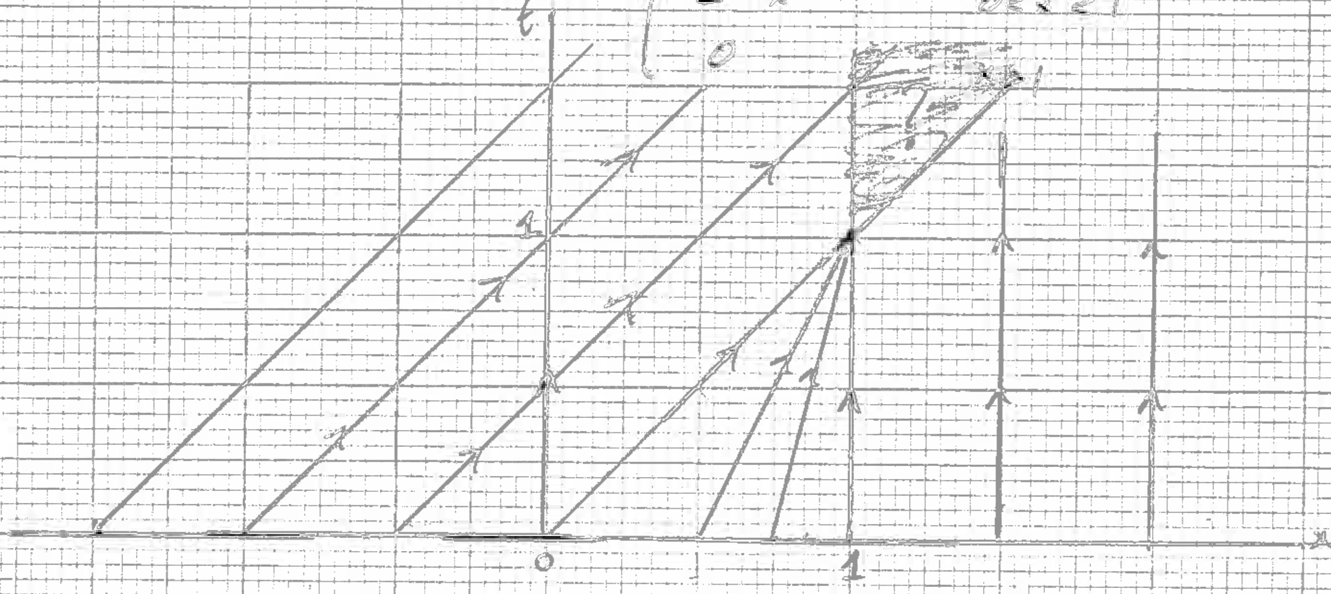


Characteristics don't fill the region $0 < x < t$ so the solution is not determined uniquely in this region from the ICs.

Shocks This is the opposite case, in which faster characteristics catch up and collide with slower ones.

Consider

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 1 - x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$



More than one characteristic enters each point in $1 < x < t$ so the solution is not uniquely determined in that region.